On the hydrodynamic and hydromagnetic stability of swirling flows

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Some general stability criteria for non-dissipative swirling flows are derived, and extended to the case of an electrically conducting fluid in the presence of axial magnetic field and current. In particular it is shown that the analogy between a rotating and a stratified fluid holds in this case, and that an important determinant of stability is a 'Richardson number' based on the analogue of the density gradient and the shear in the axial flow.

1. Introduction

In this paper we study the stability of inviscid flows between concentric cylinders, which have an axial velocity component $W(r)$, depending only on the radius r, in addition to a 'swirl' component $V(r)$ in the direction of increasing azimuth angle θ. We shall for the most part restrict attention to the case of *axisymmetric* perturbations. This is an important restriction, for if non-axisymmetric perturbations are included, new mechanisms of instability may become possible—nevertheless we feel that the study of the axisymmetric case has an interest of its own, and leads to a physical understanding of one important type of instability to which non-parallel flows in general are subject.

In the case of zero axial flow, the inviscid stability problem for axisymmetric disturbances has been well understood since Rayleigh (1916) gave his criterion that a necessary and sufficient condition for stability is that the square of the circulation, $(rV)^2$, should nowhere decrease as r increases from the inner to the outer cylinder. As Rayleigh remarked, this problem has a strong analogy with the stability of a density-stratified fluid at rest, under the action of gravity; so long as only axisymmetric perturbations are considered, one may ignore the rotation and think of the fluid as being subject to a radially outward 'gravitational field', and having a 'density' which is determined by the distribution of velocity $V(r)$. In this interpretation, Rayleigh's criterion is simply the condition that for stability the 'density' should not decrease 'downward', i.e. outward. The 'potential energy of gravity' in the analogue is in fact equal to the kinetic energy associated with the swirl velocity.

This analogy suggests that when an axial flow is also present, the effect of the

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swirl component may be analogous to the effect of density stratification (in the presence of gravity) on a parallel shear flow. In the latter case it is known that a statically stable density stratification usually tends to have a stabilizing influence on any shear instability which may be present, this effect being measured by the Richardson number; and it has been shown (Miles 1961, see also Howard 1961) that complete stability is insured if the local Richardson number everywhere exceeds $\frac{1}{4}$. Furthermore, examples are known (e.g. Drazin 1958) which show that at least in some cases $\frac{1}{4}$ is the *smallest* value of the Richardson number which will insure stability.

On the other hand, Chandrasekhar (1960*a*; 1961, § 78*b*) has considered the stability of inviscid flows with both axial and swirl components and has concluded that the effect of the swirl component is absolute, stability being determined by Rayleigh's criterion alone, without reference to the axial component. This physically rather implausible result we believe to be incorrect; the swirl component, if Rayleigh's criterion for stability is satisfied, does have a certain stabilizing influence, but one which in general is effective in producing complete stability only if it is sufficiently strong in comparison with the shear in the axial component. In § 2 we shall show that the gravitational analogy does carry over to the case of a superposed axial flow; a suitable 'Richardson number' can be defined and a sufficient condition for stability (to axisymmetric perturbations) is that it should everywhere exceed $\frac{1}{4}$. We shall also show that the complex wave speed for any unstable mode must lie in a certain semicircle, a result also known for stratified parallel shear flow.

If the fluid is taken to be a perfect electric conductor and subjected to an axial current distribution, i.e. a transverse circular magnetic field, it is known in the case of zero axial flow (Michael 1954) that the magnetic field has an effect similar to that of the swirl velocity and that Rayleigh's criterion, with a modified 'effective density', continues to hold. Here the 'potential energy of gravity' is the sum of the kinetic energy of the swirl component and the energy of the transverse magnetic field. In § 4 we shall show that the analogy with a stratified shear flow persists here also when an axial velocity is superposed, and that the 'Richardson number $\geqslant \frac{1}{4}$' and semicircle theorems hold. In §§ 5 and 6 we discuss some related results, for the case when a uniform axial magnetic field is also present, in particular a result analogous to one obtained by Velikhov (1959*a*) for non-dissipative parallel shear flow with a parallel magnetic field, and a generalization of some results of Chandrasekhar (1960*b*) and Velikhov (1959*b*), for the case of zero axial flow. In the last section we summarize the results known to us at present on the stability of non-dissipative flows involving combinations of the four components: swirl velocity, axial velocity, axial current and axial magnetic field.

2. Swirling flow

The differential equation which determines the stability to axisymmetric perturbations of an inviscid flow, between cylinders $r = R_1$ and $r = R_2$, with basic velocity $\mathbf{U} = V(r)\,\boldsymbol{\theta}_1 + W(r)\,\mathbf{k}$ is:

$$D[(W-c)^2 D_* F] - k^2(W-c)^2 F + \Phi F = 0, \tag{1}$$

where $D = d/dr$, $D_* = d/dr + 1/r$, the perturbation velocities are given by

$$u = ik(W-c)\,F\,e^{ik(z-ct)}, \tag{2}$$

$$v = -(D_*\,V)\,F\,e^{ik(z-ct)}, \tag{3}$$

$$w = -D_*[(W-c)\,F]\,e^{ik(z-ct)}, \tag{4}$$

and

$$\Phi = r^{-3}D[r^2V^2]. \tag{5}$$

A convenient reference for the derivation of this equation is Chandrasekhar (1961, § 78). The boundary conditions are that $F = 0$ on $r = R_1$ and R_2; the flow is unstable if (1) and the boundary conditions have a non-trivial solution with $\operatorname{Im} c > 0$.

The analogous equation for the stability to axisymmetric perturbations of an axial flow, $\mathbf{U} = W(r)\,\mathbf{k}$, of an inviscid fluid with basic density field $\rho_0(r)$ subject to a 'gravitational' force $g\mathbf{r}_1$ in the direction away from the axis (here we take g constant for simplicity, though there is no difficulty in allowing g to vary with r) is easily derived, and is

$$D[\rho_0(W-c)^2D_*F] - k^2\rho_0(W-c)^2F + g\rho_0'F = 0, \tag{6}$$

where the symbols have the same meaning as before, except that there is now no swirl component V and equation (3) is to be replaced by a formula for the perturbation density ρ:

$$\rho = -(D\rho_0)\,F\,e^{ik(z-ct)}. \tag{7}$$

If we make the common approximation in stratified flow stability theory of neglecting the inertial effects of density variation, (6) becomes:

$$D[(W-c)^2D_*F] - k^2(W-c)^2F + (g\rho_0'/\rho_0)\,F = 0, \tag{8}$$

i.e. equation (1) with $\Phi = g\rho_0'/\rho_0$. The two problems are thus mathematically equivalent. It is in fact possible to show that the full non-linear equations of axisymmetric swirling flow are identical with those of a stratified axial flow subject to a suitable radial gravity, inertial effects of the stratification being neglected.

Equation (8) is almost identical with the equation for stability of a parallel stratified flow subject to a uniform vertically downward gravitational field; this is obtained from (8) by replacing D_* by D, and interpreting r as the vertical co-ordinate, increasing downward. The argument given by Chandrasekhar (1961, § 78*b*) apparently shows that $\Phi > 0$ implies that (1) can have no unstable solutions, but this argument makes no essential use of the difference between D and D_*, and applies just as well to the stability problem for stratified parallel shear flow, where the conclusion would be that a statically stable density stratification *always* stabilizes *any* parallel shear flow. The argument thus proves too much. Equation (1) can, however, be studied by the techniques which have been used in the problem of parallel stratified shear flow (Howard 1961). We now carry this out.

Suppose (1) and the boundary conditions have a non-trivial solution F with $\operatorname{Im} c > 0$. Then $W-c$ does not vanish on $[R_1, R_2]$ and we can form a square root $(W-c)^{\frac{1}{2}}$ which is as smooth as W is; we assume W to be continuous and piecewise continuously differentiable. (It can be shown that the flow is always unstable for some wavelengths if W is discontinuous.) Let $G = (W-c)^{\frac{1}{2}} F$, and write (1) in terms of G. The result is

$$D[(W-c)\,D_* G] + \tfrac{1}{2}(W'/r - W'')\,G - \tfrac{1}{4}W'^2(W-c)^{-1}G - k^2(W-c)\,G + \Phi(W-c)^{-1}G = 0. \quad (9)$$

Multiplying this equation by $r\bar{G}$ and integrating over (R_1, R_2) we get

$$\int_{R_1}^{R_2} (W-c)\,[|D_* G|^2 + k^2|G|^2]\,r\,dr + \frac{1}{2}\int_{R_1}^{R_2} (rW'' - W')\,|G|^2\,r\,dr + \int_{R_1}^{R_2} (W-\bar{c})\,[\tfrac{1}{4}W'^2 - \Phi]\left|\frac{G}{W-c}\right|^2 r\,dr = 0. \quad (10)$$

The imaginary part of this equation, if $c_i \equiv \operatorname{Im} c > 0$, gives

$$\int_{R_1}^{R_2} [|D_* G|^2 + k^2|G|^2]\,r\,dr + \int_{R_1}^{R_2} [\Phi - \tfrac{1}{4}W'^2]\left|\frac{G}{W-c}\right|^2 r\,dr = 0, \quad (11)$$

which is impossible if Φ is everywhere $\geqslant \frac{1}{4}W'^2$. Thus a sufficient condition for stability is that $\Phi - \frac{1}{4}W'^2$ be everywhere non-negative. If we define a local *Richardson number* $J(y)$ by

$$J(y) \equiv \Phi/W'^2, \quad (12)$$

which equation (8) shows to be the natural definition by analogy with the variable density problem, the stability condition is $J(y) \geqslant \frac{1}{4}$ everywhere. (If $W' = 0$ somewhere, we allow $J = +\infty$.) It should perhaps be emphasized again that this applies *only* to axisymmetric perturbations. Swirling flows are sometimes unstable to non-axisymmetric perturbations even if $J(y) \geqslant \frac{1}{4}$ everywhere.

If the same procedure is applied to equation (1) instead of (9) we obtain the integral relation

$$\int_{R_1}^{R_2} (W-c)^2[|D_* F|^2 + k^2|F|^2]\,r\,dr = \int_{R_1}^{R_2} \Phi\,|F|^2\,r\,dr. \quad (13)$$

Supposing $\Phi \geqslant 0$, i.e. considering the case which is stable if $W \equiv 0$, and setting $c = c_r + ic_i$ $(c_i > 0)$, (13) gives

$$\int_{R_1}^{R_2} [(W-c_r)^2 - c_i^2]\,[|D_* F|^2 + k^2|F|^2]\,r\,dr = \int_{R_1}^{R_2} \Phi\,|F|^2\,r\,dr \geqslant 0 \quad (14)$$

and

$$\int_{R_1}^{R_2} (W-c_r)\,[|D_* F|^2 + k^2|F|^2]\,r\,dr = 0. \quad (15)$$

Setting $Q = [|D_* F|^2 + k^2|F|^2]\,r \quad (\geqslant 0 \text{ and } \not\equiv 0)$,

(14) and (15) imply

$$\int_{R_1}^{R_2} WQ\,dr = c_r\int_{R_1}^{R_2} Q\,dr \quad \text{and} \quad \int_{R_1}^{R_2} W^2Q\,dr \geqslant (c_r^2 + c_i^2)\int_{R_1}^{R_2} Q\,dr. \quad (16)$$

Suppose $a \leqslant W \leqslant b$. Then

$$0 \geqslant \int_{R_1}^{R_2} (W-a)\,(W-b)\,Q\,dr \geqslant [c_r^2 + c_i^2 - (a+b)\,c_r + ab]\int_{R_1}^{R_2} Q\,dr,$$

and thus

$$[c_r - \tfrac{1}{2}(a+b)]^2 + c_i^2 \leqslant \tfrac{1}{4}(a-b)^2. \quad (17)$$

Thus the complex wave speed c for any unstable mode must lie inside the semicircle in the upper half plane which has the range of W for diameter, if $\Phi \geqslant 0$.

In the present problem, as in the parallel stratified flow case, there are usually infinitely many neutrally stable waves, with real c lying outside the range of W. These are present when $W \equiv 0$, and are then the internal waves, in a stratified fluid, and the (axisymmetric) normal modes of a rotating column, in the present case. When $W \not\equiv 0$ such neutral modes have been proved to exist, under certain weak hypotheses about W and V by Chandrasekhar (1961, § 78*b*), in the present case, and by Miles (1961) in the stratified flow case. The semicircle theorem shows that these should be regarded as modifications by W of the (isolated) normal modes already present when $W \equiv 0$; they cannot be limits of unstable waves. However, there may be *singular* neutral modes, adjacent to unstable waves, with c in the range of W; this possibility, which was overlooked by Chandrasekhar, has been discussed at length, in the stratified flow case, by Miles (1961).

We have seen that stability is assured if the local Richardson number everywhere exceeds $\frac{1}{4}$, but the violation of this condition does not necessarily imply instability, as is known from examples in the analogous case of parallel stratified flow with vertical gravity (for instance, the plane Couette flow with exponential density is stable for all $J \geqslant 0$). On the other hand, it would be desirable to show that at least for some flows instability *does* occur for $J < \frac{1}{4}$, i.e. that as a *general* condition for stability $J \geqslant \frac{1}{4}$ cannot be improved upon. Unfortunately, examples of swirling flows for which the stability problem can be explicitly solved do not seem to be readily found. Some cases with broken-line velocity profiles can be solved, but the formulas are rather weighted down with Bessel functions, and the results are not too easily seen without numerical calculation. However, if we make the 'narrow-gap approximation', i.e. assume that $(R_2 - R_1)/R_1 \ll 1$, we may replace D_* by D in (1), and the stability equation becomes identical with that of parallel stratified flow with a vertical gravity. Any examples for this case can thus be taken over. If we furthermore assume that $R_2 - R_1$, while small compared with R_1, is large compared with the scale of variation of W (the 'wide narrow-gap approximation') we may in fact take over as examples any solutions to the stability problem for *unbounded* parallel stratified flow with gravity. The example of Drazin (1958) is perhaps the simplest, and shows that $\frac{1}{4}$ cannot be replaced by any smaller number in the stability condition '$J(y) \geqslant \frac{1}{4}$ everywhere'.

3. Remarks on the non-axisymmetric case

It is perhaps of interest to consider what can be done with these methods when the perturbations are not required to be axisymmetric. If we suppose that the radial perturbation velocity is $u(r)\,e^{i(\sigma t+kz+m\theta)}$, and that the other perturbation quantities depend similarly on t, z and θ, it is possible to eliminate all the dependent variables but u from the linearized perturbation equations and so obtain the following stability equation:

$$\gamma^2 D[SD_* u] - \{\gamma^2 + \gamma R D[S(r^{-1}D\gamma + 2mV/r^3)] - 2kVr^{-2}S[krD_* V - mDW]\}u = 0, \tag{18}$$

where $\gamma = \sigma + kW + mV/r$ and $S = r^2[m^2+k^2r^2]^{-1}$. (When $m = 0$ this reduces to (1), with $u = ik(W-c)F$, $\sigma = -kc$.) The motion is unstable if (18) possesses a non-trivial solution with $u(R_1) = u(R_2) = 0$ and $\operatorname{Im}\sigma < 0$. Supposing that we have an unstable case, set $u = H\gamma^{1-n}$, some definite branch being selected when n is not an integer as in the case discussed previously (with $n = \frac{1}{2}$) in the paragraph before equation (9). In terms of H, equation (18) becomes

$$D[S\gamma^{2(1-n)}D_* H] - \gamma^{2(1-n)}\{1+\gamma^{-1}[2mrD(SV/r^3)+nr\,D(Sr^{-1}D\gamma)] + \gamma^{-2}S[n(1-n)(D\gamma)^2 - (2kV/r^2)(kr\,D_*\,V - mDW)]\}H = 0. \quad (19)$$

$n = 1$ in (19) yields (18), with $H = u$; $n = 0$ corresponds to (1) with $H = F$, and $n = \frac{1}{2}$ to (9), with $H = G$. Taking first $n = \frac{1}{2}$, multiplying (19) by $r\bar{H}$ and integrating over (R_1, R_2) we deduce the analogue of (10). From the imaginary part of this, recalling that $\operatorname{Im}\sigma < 0$, we obtain the following analogue of (11):

$$\int_{R_1}^{R_2}[S\,|D_* H|^2 + |H|^2]\,r\,dr + \int_{R_1}^{R_2}\left[\frac{2kV}{r^2}(kr\,D_*\,V - mDW) - \tfrac{1}{4}(D\gamma)^2\right]S\left|\frac{H}{\gamma}\right|^2 r\,dr = 0. \quad (20)$$

(20) gives as a sufficient condition for stability

$$k^2\Phi - (2km/r^2)\,V\,DW - \tfrac{1}{4}[kDW + mD(V/r)]^2 \geqslant 0 \quad \text{everywhere.} \quad (21)$$

For $m = 0$ we recover the axisymmetric condition $\Phi \geqslant \frac{1}{4}(DW)^2$, but if $m \neq 0$, the condition (21) is always violated for sufficiently small k, and while this does not imply instability, it shows that no general stability criterion is obtainable in this way. (That (21) is violated for small k is immediately obvious if $D(V/r) \not\equiv 0$; in the exceptional case of rigid rotation, a second look shows that it is also sometimes violated here, since m can take either sign, unless $DW \equiv 0$, i.e. essentially unless $W \equiv 0$. Rigid rotation without axial flow *is* stable.) While (21) gives no general stability criterion it can be used to obtain bounds on the values of m/k which are possible for unstable waves. (20) can also be used to obtain an upper bound on the growth rate $\sigma_i = \operatorname{Im}\sigma$ possible for any instability, for it implies

$$\int_{R_1}^{R_2}\left\{|\gamma|^2 - S\left[\tfrac{1}{4}(D\gamma)^2 - k^2\Phi + \frac{2km}{r^2}\,VDW\right]\right\}\left|\frac{H}{\gamma}\right|^2 r\,dr \leqslant 0,$$

and since $|\gamma|^2 \geqslant \sigma_i^2$ we find

$$\sigma_i^2 \leqslant \operatorname{Max} S[\tfrac{1}{4}(D\gamma)^2 - k^2\Phi + (2km/r^2)\,VDW]$$
$$= \operatorname{Max}\frac{r^2}{(m/k)^2+r^2}\left[\tfrac{1}{4}(DW)^2 - \Phi + \frac{m}{2kr^4}DW\,.\,D(r^3V) + \left(\frac{m}{k}\right)^2.\left(D\left(\frac{V}{r}\right)\right)^2\right]. \quad (22)$$

Returning now to (19), take $n = 1$, multiply by $r\bar{H}$, and integrate over (R_1, R_2). This gives

$$\int_{R_1}^{R_2} S\,|D_* H|^2 r\,dr + \int_{R_1}^{R_2}\left\{1 + \frac{1}{\gamma}\left[2mr\,D\left(S\frac{V}{r^3}\right) + rD\left(\frac{S}{r}D\gamma\right)\right] - \frac{S}{\gamma^2}\frac{2kV}{r^2}(kr\,D_*\,V - mDW)\right\}|H|^2 r\,dr = 0, \quad (23)$$

a relation analogous to that used in the derivation of Rayleigh's theorem on the necessity of an inflexion point for instability of parallel inviscid flow (Rayleigh

1916). We have not succeeded in drawing any general conclusions from this relation, but in certain special cases analogues of Rayleigh's inflexion-point theorem can be obtained, namely in those cases in which the last term in the second integral is zero. In such a case, exactly as in the proof of Rayleigh's theorem, we can conclude that for instability to occur it is necessary that the coefficient of γ^{-1} should change sign in (R_1, R_2). We thus obtain the following results:

(*a*) $k = 0$ (two-dimensional perturbations of swirling flow). Here

$$S = r^2/m^2 \quad \text{and} \quad D\gamma = mD(V/r) = (m/r)\,D_*\,V - 2mV/r^2,$$

so

$$2mrD(SV/r^3) + rD(SR^{-1}D\gamma) = (r/m)\,DD_*\,V.$$

Since $D_*\,V$ is the vorticity of the basic swirl component, a sufficient condition for stability is that this vorticity should be a monotonic function of r on (R_1, R_2). This result is due to Rayleigh (1880), who derived it in the case $W \equiv 0$; for two-dimensional perturbations the presence of an axial flow is of course irrelevant.

(*b*) $V \equiv 0$ (non-axisymmetric perturbations of pure axial flow). Here $D\gamma = kDW$, and the quantity which (for instability) must change sign in (R_1, R_2) is $D\{r(m^2+k^2r^2)^{-1}DW\}$, i.e. a sufficient condition for stability is that $r(m^2+k^2r^2)^{-1}DW$ should be monotonic. This result is also due to Rayleigh (1892).

(*c*) $m = 0$, $D_*\,V = 0$ (axisymmetric perturbations of axial flow with a superposed irrotational vortex). Here $S = k^{-2}$, $D\gamma = kDW$, and the quantity which must change sign is $D(r^{-1}DW)$. Thus Rayleigh's condition (*b*) for pure axial flow also applies with a superposed irrotational swirl, so long as only axisymmetric perturbations are considered. In this case, of course, the results of § 2 apply; the present condition is an additional requirement supplementing the Richardson number theorem. In fact if $D_*\,V \equiv 0$, $\Phi \equiv 0$, and so $J(y) \equiv 0$. Since this is $< \frac{1}{4}$, the results of § 2 do not imply stability for any $W(y)$ other than a constant.

If we take $n = 0$ in (19) and follow the same procedure as above we obtain a relation analogous to (13) which in the axisymmetric case gives the semicircle theorem. We have not, however, been able to deduce any general results from it in the non-axisymmetric case.

The overall conclusion of this consideration of the non-axisymmetric case is thus essentially negative: the methods used to derive the Richardson number and semicircle results in the axisymmetric case reproduce the known results of Rayleigh for two-dimensional perturbations and pure axial flow, but seem to give very little more. In fact the present situation with regard to non-axisymmetric perturbations seems to be very unsatisfactory from a theoretical point of view. While there are a number of special examples, particularly in related hydromagnetic problems with no basic motion, in which the stability to non-axisymmetric perturbations has been studied by solving explicitly the differential equation, we know of no *general* results beyond those of Rayleigh cited above. An attempt has been made by Ludwieg (1960, 1961*a*) to treat the case of swirling flow, but although his argument has some plausibility (without, it seems to us, giving a mathematical proof) in the context of its original presentation for a very special kind of swirling flow, the wider extension of his stability criterion suggested in Ludwieg (1961*b*) cannot be justified, counter-examples being readily found. The attempt by Chandrasekhar (1960*c*; 1961, § 67) to show that Rayleigh's criterion $\Phi \geqslant 0$ for stability to axisymmetric perturbations of inviscid Couette flow also applies to non-axisymmetric perturbations has also been unsuccessful; the simplest counter-example is probably $V = 0$ for $R_1 \leqslant r < \frac{1}{2}(R_1+R_2)$, $V = 1$ for $\frac{1}{2}(R_1+R_2) < r \leqslant R_2$, which is unstable (for instance) to two-dimensional perturbations although the circulation is non-decreasing outwards.

4. Swirling flow of an infinitely electrically conducting fluid with a current parallel to the axis

In this section we take the basic steady flow to be $[0, V(r), W(r)]$ as before, and suppose that the basic magnetic field is $[0, H_\theta(r), 0]$, cylindrical co-ordinates being used and the circular magnetic field $H_\theta(r)$ produced by a suitable distribution of axial current. The stability problem for this flow has been considered in the case $W \equiv 0$ by Michael (1954), who has found an analogue of Rayleigh's criterion, namely, a necessary and sufficient condition for stability to axisymmetric perturbations is that the quantity

$$\Psi(r) \equiv r^{-3}D(r^2V^2) - (\mu/4\pi\rho)\, rD(H_\theta^2/r^2) \tag{24}$$

should be everywhere non-negative. Michael's result shows that when $W \equiv 0$ this hydromagnetic problem is analogous to the problem of stability of a cylindrically stratified fluid at rest, under the action of a radial gravity, just as in Rayleigh's case with $H_\theta = 0$, and this suggests that also when an axial flow is present the effect of the axial current may similarly just modify the density distribution in the analogous stratified-flow problem discussed in § 2. We now verify that this is the case by deriving the stability equation for the case $W \not\equiv 0$.

For the basic equations of non-dissipative hydromagnetics we may refer to Chandrasekhar (1961, § 80), with the dissipative terms omitted. From these equations and with the basic flow and field given above, the usual process of linearization leads, with a (z,t)-dependence of the form $e^{ik(z-ct)}$, to the following system of perturbation equations (axisymmetric perturbations only are considered, and lowercase letters are used for the perturbation velocity and field components)

$$ik(W-c)\,u - 2Vv/r + (\mu/2\pi\rho)\, H_\theta h_\theta/r = -Dp, \tag{25}$$

$$ik(W-c)\,v + uD_* V = 0, \tag{26}$$

$$ik(W-c)\,w + uDW = -ikp, \tag{27}$$

$$ik(W-c)\,h_\theta + ruD(H_\theta/r) = 0, \tag{28}$$

$$D_* u + ikw = 0, \tag{29}$$

$$h_r = h_z = 0 \tag{30}$$

(p is a suitable perturbation pressure, including magnetic effects). If we set

$$u = ik(W-c)\,F \tag{31}$$

(assuming that $\operatorname{Im} c > 0$), we find

$$v = -(D_* V)\,F, \tag{32}$$

$$w = -D_*[(W-c)\,F], \tag{33}$$

$$h_\theta = -rD(H_\theta/r)\,F, \tag{34}$$

$$p = (W-c)^2 D_* F. \tag{35}$$

(Since (31) gives the boundary condition $F = 0$ on R_1 and R_2, (34) shows that for consistency we should take the cylindrical walls to be non-conducting.) Using these in (25), we get

$$D[(W-c)^2 D_* F] - k^2(W-c)^2 F + [\Phi - (\mu/4\pi\rho)\, rD(H_\theta^2/r^2)]\, F = 0. \quad (36)$$

This equation is identical with (1) except that Rayleigh's discriminant Φ is replaced by Michael's Ψ, defined by (24). The results of § 2 can thus be taken over at once, simply by replacing Φ by Ψ, i.e.

(*a*) A sufficient condition for stability to axisymmetric perturbations of a non-dissipative swirling flow with axial current is

$$\Psi \geqslant \tfrac{1}{4}W'^2 \quad \text{everywhere}, \quad (37)$$

or that the *local Richardson number* $J(y) \equiv \Psi/W'^2$ should nowhere be less than $\frac{1}{4}$.

(*b*) If $\Psi \geqslant 0$, i.e. if the flow is stable according to Michael in the absence of axial flow, then the complex wave speed c of any instability which might occur when axial flow is present must lie inside the semicircle in the upper half plane which has the range of W for diameter.

We note one more result, an upper bound on the growth rate of any instability, which is obtained as in the case of (22) above:

(*c*)
$$k^2 c_i^2 \leqslant \mathrm{Max}\,[\tfrac{1}{4}W'^2 - \Psi]. \quad (38)$$

Thus, at least so far as stability to axisymmetric perturbations is concerned, the circular magnetic field, like the swirl component V, has an effect analogous to a density stratification in a radial gravitational field, and the 'effective Richardson numbers' of circular field and swirl are additive.

A special case of some interest arises when the applied current density J_0 is independent of r, so that $H_\theta = 2\pi J_0 r$. In this case $\Psi \equiv \Phi$ and the condition for stability is unaffected by the current. In fact, equation (28) shows that $h_\theta \equiv 0$, and not only the sufficient condition for stability, but in fact the full stability problem is unaffected by the presence of the axial current, the only effect of the electromagnetic stress being to modify the basic pressure field. As remarked by Michael (1954) in the case $W \equiv 0$, this is physically clear because H_θ/r is constant following particles, and in the case of uniform current is initially the same for all particles, so that whatever axisymmetric motion may be imposed on the fluid, the magnetic field strength remains the same at every point. This conclusion about the irrelevance of a uniform current to instability with respect to axisymmetric perturbations also holds in fact when viscosity and finite conductivity are allowed for (with non-conducting walls); in this case it can also be shown that any unstable wave must have $h_\theta = 0$ (physically, the magnetic field gives no opportunity for magnetic diffusion) and that H_θ then disappears from the equations. However, the uniform axial current may have an effect on the decay rate of stable perturbations, which need not necessarily have $h_\theta \equiv 0$.

5. Axial flow with uniform axial magnetic field

In this section we suppose $V = H_\theta \equiv 0$, and study the effect on stability to axisymmetric perturbations of a uniform *axial* magnetic field H_0. This problem, allowing also for finite electrical conductivity, has been discussed recently by

Jain (1961). He has concluded that all inviscid axial flows with uniform axial magnetic field are stable with respect to axisymmetric perturbations, but this conclusion is unwarranted. It arises from the fact that he tacitly assumed in his analysis that the azimuthal velocity and magnetic field components (v and h_θ) were not identically zero. However, the perturbation equations may have solutions with $v = h_\theta \equiv 0$, and in fact his argument essentially proves only that if there is to be any (axisymmetric) unstable wave, v and h_θ must be zero. It does not exclude instability if they *are* zero. (That $v = h_\theta = 0$ for any axisymmetric instability can also be shown when viscosity is allowed for too, by the same argument.)

The perturbation equations for the present problem, derived in the usual way with $e^{ik(z-ct)}$ (z, t)-dependence are:

$$ik(W-c)\,u-(\mu H_0/4\pi\rho)\,ikh_r = -Dp, \tag{39}$$

$$ik(W-c)\,w+uDW-(\mu H_0/4\pi\rho)\,ikh_z = -ikp, \tag{40}$$

$$ik(W-c)\,h_r = H_0\,iku, \tag{41}$$

$$ik(W-c)\,h_z = h_r DW+ikH_0 w, \tag{42}$$

$$D_* u+ikw = 0, \tag{43}$$

$$D_* h_r+ikh_z = 0, \tag{44}$$

and

$$v = h_\theta = 0. \tag{45}$$

(The notation is as in § 4; (45), as mentioned above, follows from Jain's argument, assuming $c_i > 0$, even with viscosity and finite conductivity, but in the present case it is an immediate consequence of the perturbation equations.) We now introduce F as before,

$$u = ik(W-c)\,F, \tag{46}$$

and then find from (40)–(44) (only four of which are independent),

$$w = -D_*[(W-c)\,F], \tag{47}$$

$$h_r = ikH_0 F, \tag{48}$$

$$h_z = -H_0 D_* F, \tag{49}$$

$$p = [(W-c)^2-(\mu H_0^2/4\pi\rho)]\,D_*\,\mathrm{F}. \tag{50}$$

Using these equations in (39), we finally obtain the stability equation for F:

$$D\{[(W-c)^2-V_A^2]\,D_* F\}-k^2[(W-c)^2-V_A^2]\,F = 0, \tag{51}$$

where $V_A = (\mu H_0^2/4\pi\rho)^{\frac{1}{2}}$ is the Alfvén velocity.

In the present case we do not have a physical mechanism analogous to density stratification with gravity, and we have not found any analogue of the Richardson number theorem. However, we can deduce from (51) an analogue of the semicircle theorem, as follows: multiply (51) by $r\bar{F}$ and integrate over (R_1, R_2). This gives

$$\int_{R_1}^{R_2}[(W-c)^2-V_A^2]\,[|D_* F|^2+k^2\,|F|^2]\,r\,dr = 0.$$

Taking $c = c_r + ic_i$, setting $Q = [|D_* F|^2 + k^2 |F|^2] r$, and separating real and imaginary parts we obtain

$$\int_{R_1}^{R_2} [(W-c_r)^2 - c_i^2 - V_A^2] Q \, dr = 0, \tag{52}$$

$$c_i \int_{R_1}^{R_2} (W-c_r) Q \, dr = 0. \tag{53}$$

As in § 2, these equations imply ($c_i > 0$)

$$\int_{R_1}^{R_2} WQ \, dr = c_r \int_{R_1}^{R_2} Q \, dr \tag{54}$$

and
$$\int_{R_1}^{R_2} W^2 Q \, dr = (c_r^2 + c_i^2 + V_A^2) \int_{R_1}^{R_2} Q \, dr. \tag{55}$$

Supposing that $a \leqslant W \leqslant b$ we thus find

$$0 \geqslant \int_{R_1}^{R_2} (W-a)(W-b) Q \, dr = [\{c_r - \tfrac{1}{2}(a+b)\}^2 + c_i^2 + V_A^2 - \{\tfrac{1}{2}(a-b)\}^2] \int_{R_1}^{R_2} Q \, dr$$

and thus
$$\{c_r - \tfrac{1}{2}(a+b)\}^2 + c_i^2 \leqslant \tfrac{1}{4}(a-b)^2 - V_A^2. \tag{56}$$

The complex wave speed c for any unstable wave must lie in a semicircle in the upper half plane, concentric with the semicircle having the range of W for diameter, and of radius $[\{\frac{1}{2}(a-b)\}^2 - V_A^2]^{\frac{1}{2}}$. This result is in a sense stronger than the previous semicircle theorem, for by itself it gives:

A sufficient condition for stability to axisymmetric perturbations of an axial flow with uniform axial magnetic field is that the Alfvén speed of the axial field should exceed half the maximum velocity difference.

This result is the axisymmetric analogue of the two-dimensional result of Velikhov (1959*a*). It may be given a certain physical rationalization by stating it in the following way: instability is possible only if there are two radii, r_1 and r_2, in the flow such that $W(r_1) + V_A = W(r_2) - V_A$, i.e. such that Alfvén waves moving in opposite directions, relative to the two local flow velocities, do not move relative to each other. Of course such localized Alfvén waves are not really possible except for very short wavelengths, but one may perhaps loosely think of the instability as arising from an interaction in this way. If the magnetic field is strong enough, the two waves cannot move at the same absolute speed and do not reinforce each other. We offer this interpretation only as a simple way of remembering the stability condition, and do not insist on its physical significance.

To give an example illustrating this stability condition we may make use of the 'wide narrow-gap approximation' and take over Drazin's (1960) example of an unbounded jet having $W = W_0$ in a finite interval, and $W = 0$ elsewhere. Drazin found, in the present case of a single fluid and infinite magnetic Reynolds number, that this flow is stable when, and only when, $\mu H_0^2/4\pi\rho \geqslant (\frac{1}{2}W_0)^2$, i.e. instability occurs exactly at the point at which the general stability criterion ceases to forbid it. The axisymmetric analogue of Drazin's jet has been studied by Michael (1962), the smallest magnetic field which will stabilize axisymmetric perturbations of all wavelengths again being found to be given exactly by the semicircle theorem. (This 'exactness' in these examples is no doubt a feature of the discontinuous profiles, and is not to be expected in general.)

6. Non-dissipative Couette flow with axial current and uniform axial magnetic field

In this section we take for the basic flow $\mathbf{U} = [0, V(r), 0]$, and for the basic magnetic field $\mathbf{H} = [0, H_\theta(r), H_0]$, in cylindrical co-ordinates. The stability equation in this case, with the same notation as before, is (cf. Chandrasekhar 1961, § 84)

$$(c^2 - V_A^2)(DD_* - k^2) F + \left[\Psi - 4\left(\frac{V^2}{r^2} + \frac{V_A^2}{r^2}\frac{H_\theta^2}{H_0^2}\right)\right] F + \frac{4c}{c^2 - V_A^2}\left[\left\{\frac{V^2}{r^2} + \frac{V_A^2}{r^2}\cdot\left(\frac{H_\theta}{H_0}\right)^2\right\} c + \frac{2V_A^2}{r^2}\cdot V\frac{H_\theta}{H_0}\right] F = 0. \qquad (57)$$

The case $H_\theta \equiv 0$ was considered by Velikhov (1959*b*) and Chandrasekhar (1960*b*), while the case $V \equiv 0$ was given by Chandrasekhar (1961, § 84*a*). In both these special cases it is easily seen that c^2 is real; in the general case this is not necessarily so, and this requires some modification of the arguments of Velikhov and Chandrasekhar. Multiplication of (57) by $r\bar{F}$ and integration over (R_1, R_2) gives

$$(c^2 - V_A^2)\int_{R_1}^{R_2} [|D_* F|^2 + k^2 |F|^2]\, r\, dr - \int_{R_1}^{R_2}\left[\Psi - 4\left(\frac{V^2}{r^2} + \frac{V_A^2}{r^2}\frac{H_\theta^2}{H_0^2}\right)\right] |F|^2 r\, dr - 2c\int_{R_1}^{R_2}\left\{\left[\frac{V^2}{r^2} + \frac{V_A^2}{r^2}\left(\frac{H_\theta}{H_0}\right)^2\right]\left[\frac{1}{c - V_A} + \frac{1}{c + V_A}\right] + \frac{2V_A V}{r^2}\frac{H_\theta}{H_0}\left[\frac{1}{c - V_A} + \frac{1}{c + V_A}\right]\right\} |F|^2 r\, dr = 0,$$

or

$$(c^2 - V_A^2)\int_{R_1}^{R_2} [|D_* F|^2 + k^2 |F|^2]\, r\, dr - \int_{R_1}^{R_2}\left[\Psi - 4\left(\frac{V^2}{r^2} + \frac{V_A^2}{r^2}\frac{H_\theta^2}{H_0^2}\right)\right] |F|^2 r\, dr - 2c\int_{R_1}^{R_2}\left\{\frac{\bar{c} - V_A}{|c - V_A|^2}\left(\frac{V}{r} + \frac{V_A}{r}\frac{H_\theta}{H_0}\right)^2 + \frac{\bar{c} + V_A}{|c + V_A|^2}\left(\frac{V}{r} - \frac{V_A}{r}\frac{H_\theta}{H_0}\right)^2\right\} |F|^2 r\, dr = 0. \qquad (58)$$

Now multiply (58) by $\bar{c}$ and take the imaginary part

$$c_i(|c|^2 + V_A^2)\int_{R_1}^{R_2} [|D_* F|^2 + k^2 |F|^2]\, r\, dr + c_i\int_{R_1}^{R_2}\left[\Psi - 4\left(\frac{V^2}{r^2} + \frac{V_A^2}{r^2}\frac{H_\theta^2}{H_0^2}\right)\right] |F|^2 r\, dr + 2c_i|c|^2\int_{R_1}^{R_2}\left\{\frac{1}{|c - V_A|^2}\left(\frac{V}{r} + \frac{V_A}{r}\frac{H_\theta}{H_0}\right)^2 + \frac{1}{|c + V_A|^2}\left(\frac{V}{r} - \frac{V_A}{r}\frac{H_\theta}{H_0}\right)^2\right\} |F|^2 r\, dr = 0. \qquad (59)$$

This is evidently impossible with $c_i > 0$ if the integrand of the second integral is everywhere non-negative, so a sufficient condition for stability is that

$$\Psi - \frac{4}{r^2}\left(V^2 + V_A^2\frac{H_\theta^2}{H_0^2}\right) \equiv \Psi - \frac{4}{r^2}\left(V^2 + \frac{\mu H_\theta^2}{4\pi\rho}\right) \geqslant 0 \quad \text{everywhere.} \qquad (60)$$

In the special cases $V \equiv 0$ or $H_\theta \equiv 0$ this reduces to the conditions given by Chandrasekhar and Velikhov.

As in the two special cases, this condition does not involve H_0, and consequently exhibits the same apparent paradox that the condition (60) for stability with $H_0 > 0$, however small, does not reduce to Michael's condition $\Psi > 0$ for $H_0 \to 0$. In fact, if $\Psi > 0$ but (60) is not satisfied, we may expect an axial magnetic field to have a destabilizing effect, however small it is, though it will also stabilize the flow for any fixed wave number if H_0 is large enough. The physical reason for this destabilizing effect was clearly explained by Veliklov (1959*b*), in the case $H_\theta \equiv 0$, who also pointed out that the apparent 'discontinuity' in the stability condition at $H_0 = 0$ would undoubtedly be removed if dissipative effects were taken into account. That the 'discontinuity' is only apparent can in fact be seen already in the non-dissipative theory, for we can show that if $\Psi \geqslant 0$ then, for any fixed k, if we have an instability its growth rate kc_i must approach zero as $H_0 \to 0$. To see this, note that if $c_i > 0$ and $\Psi \geqslant 0$, (59) implies

$$(|c|^2+V_A^2)\int_{R_1}^{R_2}[|D_* F|^2+k^2|F|^2]\,r\,dr-4\int_{R_1}^{R_2}\frac{1}{r^2}\left(V^2+\frac{\mu H_\theta^2}{4\pi\rho}\right)|F|^2 r\,dr$$
$$+2|c|^2\int_{R_1}^{R_2}\left\{\frac{1}{|c-V_A|^2}\left(\frac{V}{r}+\frac{V_A}{r}\frac{H_\theta}{H_0}\right)^2+\frac{1}{|c+V_A|^2}\left(\frac{V}{r}-\frac{V_A}{r}\frac{H_\theta}{H_0}\right)^2\right\}|F|^2 r\,dr \leqslant 0. \quad (61)$$

If we suppose that $V_A \to 0$ in this relation, and that c_i does *not* approach zero, the third integral is easily seen in the limit just to cancel the second, thus producing a contradiction; consequently c_i and so kc_i must approach zero as $H_0 \to 0$, if $\Psi \geqslant 0$, and we recover Michael's condition.

That a sufficiently strong axial field will stabilize at any fixed wave number can also be readily seen from (61), for it implies

$$\int_{R_1}^{R_2}\left[V_A^2 k^2-\frac{4}{r^2}\left(V^2+\frac{\mu H_\theta^2}{4\pi\rho}\right)\right]|F|^2 r\,dr < 0,$$

and this is impossible if V_A is large enough. If R_2-R_1 is finite this conclusion holds uniformly for all wave numbers, for then $\int_{R_1}^{R_2}|D_* F|^2 r\,dr$ is bounded from below by a constant, say k_1, times $\int_{R_1}^{R_2}|F|^2 r\,dr$ and we have

$$\int_{R_1}^{R_2}\left[V_A^2(k_1+k^2)-\frac{4}{r^2}\left(V^2+\frac{\mu H_\theta^2}{4\pi\rho}\right)\right]|F|^2 r\,dr < 0.$$

7. Summary

We have not been able to obtain stability criteria for flows in which all four components, V, W, H_θ, H_0 are simultaneously present. When one of the four is zero, we have results in two of the four possibilities (only axisymmetric perturbations are considered):

(*a*) $H_0 = 0$ (§ 4). Stability is assured if $\Psi \geqslant \frac{1}{4}W'^2$ everywhere. The semicircle theorem holds for unstable waves if $\Psi \geqslant 0$.

(*b*) $W \equiv 0$ (§ 6). Stability is assured if $\Psi - 4r^{-2}(V^2+\mu H_\theta^2/4\pi\rho) \geqslant 0$ everywhere when $H_0 \neq 0$ and if $\Psi \geqslant 0$ everywhere when $H_0 = 0$. When two (or more) of the four components are absent the only case which is not included in the above is (*c*) $V \equiv H_\theta \equiv 0$ (§ 5). The modified semicircle theorem holds: the complex c of any unstable wave must lie in the semicircle in the upper half plane

$$\{c_r - \tfrac{1}{2}(a+b)\}^2 + c_i^2 \leqslant \tfrac{1}{4}(a-b)^2 - V_A^2,$$

where $a \leqslant W \leqslant b$, and stability is assured if $V_A > \frac{1}{2}(b-a)$. The cases which remain open are: (*d*) H_θ alone absent, (*e*) V alone absent, and, of course, (*f*) all four present.

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